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Generalized Hall Acceleration

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The operational characteristics of electric propulsion devices which utilize the Hall effect have been generalized. The electrostatic acceleration is enhanced by a thermoelectric effect (the effect of electron pressure gradient in a generalized Ohm's law); an ion kinetic energy can be higher than that associated with the electrostatic potential. Depending on the extent of this effect, there exists two acceleration modes; an electrostatic one and an electrostatic/electrothermal hybrid one, the latter of which has a low-voltage characteristic.

Nomenclature

A = cross-sectional area
$\mathbf{B}$ = magnetic field
$\mathbf{E}$ = electric field
$\mathbf{E}'$ = effective electric field, Eq. (2)
e = charge of electron
F = thrust
J_d = discharge current
J_e = electron current
J_i = ion current
J_p = propellant flow rate in current units
$\mathbf{j}$ = current density
k = Boltzmann's constant
L = axial length
$\mathbf{l}$ = vector along electric field
M = ion mass
$\dot{m}$ = total mass flow rate
$\dot{m}_i$ = ion mass flow rate
p_e = electron pressure
R = radius
r, θ, z = cylindrical coordinates
T_e = electron temperature
u_{ex} = exhaust velocity
V = voltage applied in acceleration region
V_{eff} = effective voltage, Eq. (14)
V_{ion} = ionization potential
V_0 = volume
β = $1/(en_e)$
γ_E = (kinetic/potential) energy ratio, Eq. (19)
γ_I = normalized ionization potential, Eq. (22)
γ_T = normalized electron enthalpy, Eq. (21)
η_a = acceleration efficiency, Eq. (17)
η_{pt} = potential efficiency, Eq. (16)
η_t = thrust efficiency, Eq. (27)
η_u = propellant utilization efficiency, Eq. (18)
σ = electrical conductivity without magnetic field
χ = normalized thermoelectromotive potential, Eq. (24)
$\omega_e\tau_e$ = electron Hall parameter

Subscripts
e = electron
i = ion
$\bar{\ }$ = representative value (directly over quantity)

I. Introduction

AMONG high-powered electric propulsion devices, according to the usual categorization, there seems to exist two promising candidates which can fulfill high I_{sp} and high thrust density requirements; a Hall current thruster (or a closed-drift thruster[1,2]) and an MPD thruster (or a butt-end Hall thruster[3,4]).

Basically, a Hall current thruster usually produces thrust by the interaction of an applied radial magnetic field with an induced azimuthal current. The azimuthal current is generated basically by $\mathbf{E} \times \mathbf{B}$ drift of electrons. Since a simple analysis shows that the ion kinetic energy in principle equals the electrostatic energy associated with the discharge voltage, a thruster of this type is usually categorized as an electrostatic thruster.

In the case of MPD thrusters, in the input power range of tens of kilowatts, an external magnetic field needs to be applied in the acceleration region in order to improve thruster performance.[5,6] From the thrust formula for an applied-field MPD thruster derived by Sasoh and Arakawa,[5] "generalized Hall acceleration" is found to be suitable for high I_{sp} operation. Here, generalized Hall acceleration denotes one in which electromagnetic force is produced by the interaction between the applied magnetic field and azimuthal current induced both by Hall effect (the effect of $\mathbf{E} \times \mathbf{B}$ drift of the electrons) and by "thermoelectric effect" ("diamagnetic effect" in Ref. 5; in this article, the thermoelectric effect is defined as an effect that electron pressure gradient acts as an equivalent electric field, discussed in detail later).[5–8]

Although thrusters of these types utilize the same acceleration principle, they have been studied almost separately—no general relations have been obtained. The purpose of this study is to theoretically derive general relations which characterize, and with which one can optimize, the above-mentioned thruster operations.

II. Thruster Geometries

Here, one deals with two types of simplified thruster geometries, which are shown in Figs. 1 and 2.

The thruster geometry of "type A" is shown in Fig. 1. It has an acceleration channel in the axial direction. The electric field and the magnetic field are applied in the axial and radial directions, respectively. The interaction between axial discharge current and the magnetic field results in an induced azimuthal current, which in turn, interacts with the applied magnetic field, producing an axial force, i.e., thrust.

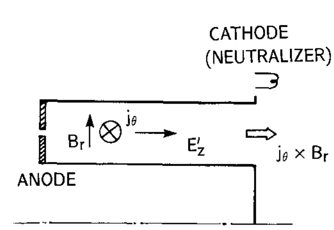

Fig. 1 Simplified thruster geometry of type A.

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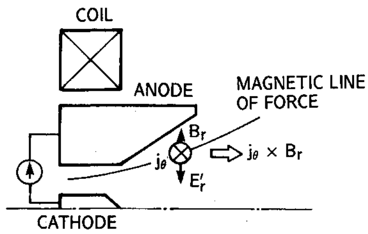

Fig. 2 Simplified thruster geometry of type B.

The thruster geometry of "type B" is shown in Fig. 2. Here, one assumes a slowly diverging magnetic field. An azimuthal current is induced mainly by the interaction between the radial discharge current and the axial magnetic field. The axial force is produced by the interaction between the induced azimuthal current and the radial component of the magnetic field. Although some other acceleration mechanisms may exist in the thruster of this type, Sasoh and Arakawa[5,6] found operating conditions under which the generalized Hall acceleration is dominant. In this study, only this acceleration mechanism is taken into account.

III. Generation of Azimuthal Current

In this study on current generation, the following three assumptions are made: 1) the current density is determined by a generalized Ohm's law; anomalous electron diffusion does not occur; 2) the magnetic Reynolds number is much smaller than unity; and 3) $\omega_e\tau_e$ is much larger than unity. Although under some operating conditions the dependence of current density on the magnetic field may quantitatively deviate from that derived based on assumption 1, this assumption does not violate the generality of the acceleration principles which are discussed in this article. At high I_{sp} operation regimes, the magnetic Reynolds number can be comparable with unity; the effect of back electromotive force is not negligible. Under such a condition, a correction term which expresses this effect ($\mathbf{u} \times \mathbf{B}$ term) in a generalized Ohm's law is not negligible.[6] However, again assumption 2 does not violate the above generality. Based on these discussions, the above assumptions are introduced for simplicity.

The generalized Ohm's law is

$$\mathbf{j} = \frac{\sigma}{1 + (\omega_e\tau_e)^2}\left[\mathbf{E}' - (\omega_e\tau_e)\frac{\mathbf{E}' \times \mathbf{B}}{B} + (\omega_e\tau_e)^2\frac{(\mathbf{E}'\cdot\mathbf{B})\mathbf{B}}{B^2}\right] \tag{1}$$

Under assumption 2, $\mathbf{E}'$ in Eq. (1) becomes

$$\mathbf{E}' = \mathbf{E} + \beta\nabla p_e \tag{2}$$

The second term of Eq. (2) represents the thermoelectric effect; the electron pressure gradient acts as an effective electric field. Applying assumption 3, and neglecting the $\mathbf{E}'\cdot\mathbf{B}$ term, the azimuthal currents are approximately expressed by

$$j_\theta \simeq \frac{1}{\beta}\frac{E'}{B} \tag{3}$$

Here, type A

$$E' = E'_z, \qquad B = B_r \tag{4}$$

type B

$$E' = E'_r, \qquad B = B_z \tag{5}$$

IV. Thrust Production

The thrust produced by the interaction between the azimuthal current and the applied magnetic field is calculated by integrating the interaction force over the acceleration region:

$$F = \dot{m}_i u_{ex} = \int j_\theta B_r \, dV_0 \tag{6}$$

Here, the thrust is assumed to equal the momentum flux of the exhaust ion flow.

A. Type A

Using representative values, the integral in Eq. (6) becomes[5]

$$\int j_\theta B_r \, dV_0 = \frac{1}{\beta}\frac{\bar{E}'_z}{\bar{B}_r}\bar{B}_r\bar{A}\bar{L} \tag{7}$$

$$\dot{m}_i = M\bar{n}_e\bar{u}\bar{A} \tag{8}$$

where $\bar{A}$ and $\bar{L}$ denote the representative values of the cross-sectional area and acceleration length, respectively. In order to satisfy both energy and momentum conservations, in principle, the relation $\bar{u} = u_{ex}/2$ must be satisfied.[5] However, taking nonuniform current distribution into account, the above relation is modified to $\bar{u} = u_{ex}/(2\xi)$, where ξ is of the order of unity. With this modified relation, and from Eqs.(6–8)

$$Mu_{ex}^2/(2e) = \xi\bar{E}'_z\bar{L} \tag{9}$$

B. Type B

In the same way, the thrust production equation of type B is expressed as follows:

$$\int j_\theta B_r \, dV_0 = \frac{1}{\beta}\frac{\bar{E}'_r}{\bar{B}_z}\bar{B}_r\pi\bar{R}^2\bar{L} \tag{10}$$

$$\dot{m}_i = M\bar{n}_e\bar{u}\pi\bar{R}^2 \tag{11}$$

Here again, $\bar{u} = u_{ex}/(2\xi')(\xi' \simeq 1)$. If the approximation $(\bar{B}_r/\bar{B}_z)/(\bar{R}/\bar{L}) = 1$ is made, Eq. (10) is transformed such that

$$Mu_{ex}^2/(2e) = \xi'\bar{E}'_r\bar{R} \tag{12}$$

C. General Expression

In summary, using the common notation ξ, the general expression of the thrust production is

$$Mu_{ex}^2/(2e) = \xi V_{eff} \tag{13}$$

$$V_{eff} = \int \mathbf{E}'\cdot d\mathbf{l} \tag{14}$$

In Eq. (14), the integration is to be done through a current path from an anode to a cathode.

V. Energy Equation

If neither energy loss to the electrodes directly due to sheath drop nor energy loss to insulators is taken into account, the discharge energy input into the acceleration region is converted into ion kinetic energy, electron enthalpy, and ionization energy. This relation is expressed by

$$J_d V = \frac{Mu_{ex}^2}{2e}J_i + \frac{5}{2}\frac{kT_e}{e}J_e + V_{ion}J_i \tag{15}$$

Note that $V = \int \mathbf{E}\cdot d\mathbf{l}$. Defining η_{pt}, V is related to the total discharge voltage V_d by

$$\eta_{pt} = V/V_d \tag{16}$$

Table 1 Examples of thruster operating conditions

Thruster	Propellant	J_p, A	J_d, A	V_d, V	η_a	η_u	γ_E[a]	V,[b] V	χ[b,c]	$\frac{5}{2}kT_e$,[b] eV	V_{ion}, eV
Example 1 (type A)[9]	A_r	2.0	1.5	180	0.31	0.23	0.84	63	0.12 ± 0.02	30 ± 5	15.8
Example 2 (type A)[11]	X_e	2.2	3.1	200	0.68	0.95	0.63	110	0.12 ± 0.03	38 ± 8	12.1
Example 3 (type B)[6]	H_2	86	200	39.4	0.32	0.73	1.32	16	0.88 ± 0.06	12 ± 1	15.8[d]

[a]Equation (19). [b]Measured by electrostatic probes. [c]Equation (24). [d]The dissociation energy per atom is added to the ionization energy.

Table 2 Summary of acceleration modes

T_e level, $\frac{5}{2}kT_e/(eV_{ion})$	Optimum η_a	Optimum χ	Acceleration mode	Characteristics
High, >1	~1	Small	Electrostatic	
Low, <1	~0	Large	Electrostatic/electrothermal hybrid	Low-voltage operation

η_a and η_u are respectively defined as follows[9]:

$$\eta_a = J_i/J_d \tag{17}$$

$$\eta_u = J_i/J_p \tag{18}$$

γ_E is defined by

$$\gamma_E \equiv Mu_{ex}^2/(2eV) \tag{19}$$

Using the above parameters, Eq. (15) is transformed into

$$\gamma_E\eta_a + \gamma_T(1 - \eta_a) + \gamma_I\eta_a = 1 \tag{20}$$

normalized electron enthalpy

$$\gamma_T \equiv \tfrac{5}{2}kT_e/(eV) \tag{21}$$

normalized ionization potential

$$\gamma_I \equiv V_{ion}/V \tag{22}$$

Integrating Eq. (2) yields

$$V_{eff} = (1 + \chi)V \tag{23}$$

normalized thermoelectromotive potential

$$\chi \equiv \int \beta\nabla p_e \cdot \frac{dl}{V} \tag{24}$$

Here, χ can be either positive or negative depending on the electron pressure distribution in the acceleration region. In many cases, χ has a positive value. However, e.g., when one employs a double-stage discharge thruster,[10] χ can have a negative value.

VI. Optimization of Operating Condition

Combining Eqs. (13), (19), and (23) yields the (kinetic/potential) energy ratio, such that

$$\gamma_E = (1 + \chi)\xi \tag{25}$$

Note here that through the thermoelectric effect γ_E can be larger than unity. When χ has a large value, the electron thermal energy can act as a "buffer" for the input discharge energy to the ion kinetic energy; some of the discharge energy once input into the electron thermal energy is finally converted into ion kinetic energy.

The examples of some thruster operating conditions are tabulated in Table 1. In examples 1 and 2, χ is much smaller than unity, implying that the ion is accelerated mainly through the electrostatic potential—"electrostatic acceleration mode."

In example 3, χ is of the order of unity; the thermoelectromotive potential is comparable with the electrostatic potential. Such thruster operation may be referred to as "electrostatic/electrothermal hybrid acceleration." In this acceleration mode the ion kinetic energy can be larger than that associated with the electrostatic potential. In other words, such a thruster can run with a low voltage.

Substituting Eq. (25) for Eq. (20), η_a is expressed also as a function of χ

$$\eta_a = \frac{1 - \gamma_T}{(1 + \chi)\xi - \gamma_T + \gamma_I} \tag{26}$$

Usually, both the denominator and numerator of Eq. (26) are positive. Therefore, if γ_T, γ_I, and ξ are assumed constant, η_a becomes a decreasing function of χ. The effects of B on the operational characteristics are not fully discussed here. However, it has experimentally been found that B (or $\int \mathbf{B}\cdot d\mathbf{l}$) is the key parameter for the determination of η_a^2 (and then χ); increasing B leads to an increase in η_a, and therefore, to a decrease in χ.

η_t[7,9] is defined and transformed, using Eq. (20), such that

$$\eta_t \equiv \frac{F^2}{2\dot{m}J_dV_d} = \eta_a\eta_u\eta_{pt}\gamma_E = \eta_u\eta_{pt}[(\gamma_T - \gamma_I)\eta_a + 1 - \gamma_T] \tag{27}$$

With constant η_u, η_{pt}, γ_T, and γ_I, η_t becomes maximum when

$$\eta_a \to 1 \qquad (\gamma_T > \gamma_I) \tag{28}$$

$$\eta_a \to 0 \qquad (\gamma_T < \gamma_I) \tag{29}$$

In examples 1 and 2, condition (28) holds. It is found from Eqs. (26) and (27) that when the electron enthalpy is higher than the ionization energy, electrostatic acceleration, i.e., a small χ, as seen in these examples, is favorable. To the contrary, in example 3 condition (29) holds. In this case, electrostatic/electromagnetic hybrid acceleration, i.e., a large χ, is preferable.

In Eq. (15), the second and third terms in the right side represent the energy loss to the anode and the frozen flow loss, respectively. In order to maximize the thrust efficiency, the sum of these terms should be minimized under the condition $J_e + J_i = J_d =$ const. When the electron enthalpy is higher than the ionization energy, the sum is mathematically minimized at $J_e = 0(J_i = J_d)$. When, on the other hand, the electron enthalpy is lower than the ionization energy, the sum is mathematically minimized at $J_e = J_d(J_i = 0)$. Although physically, neither $J_e = 0$ nor $J_i = 0$ is possible, the actual optimum condition may lie in the vicinity of the above condition. The optimum operating conditions are summarized in Table 2.

VII. Conclusions

The operation characteristics of electric propulsion devices which utilize the Hall acceleration principle have been generalized using common equations [Eqs. (3), (13), and (27)]. The induced azimuthal current and the thrust can be increased by the thermoelectric effect; some electron thermal energy can be converted into ion kinetic energy through this effect. The contribution of this effect to thrust production is estimated by a parameter χ. At large χ, ion kinetic energy can be larger than that associated with electrostatic potential. Such acceleration may be referred to as electrostatic/electrothermal hybrid acceleration. The condition which increases the thrust efficiency depends on the electron enthalpy (or temperature) level. On one hand, at an electron enthalpy level lower than the ionization energy, electrostatic/electromagnetic hybrid acceleration is favorable. On the other hand, at an electron enthalpy level higher than the ionization energy, electrostatic acceleration is favorable. The present analysis has shown the possibility of low-voltage, high-efficiency operation in the former acceleration mode.

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